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LETTER TO THE EDITOR

A two-parameter quantization of osp(4/2)

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Abstract. A two-parameter deformation of the universal algebra of osp(4/2) is carried out, yielding a Z_2-graded Hopf algebra with a bijective antipode. This Hopf algebra depends on the extra parameter in both its algebraic and co-algebraic structures, and also admits non-trivial finite-dimensional irreps at arbitrary deformation parameters.

Multiparameter quantizations of Lie algebras and superalgebras have been studied by a number of authors [1–5]. In [2] and [3], consistent deformations of the universal enveloping algebras of gl(n) and other Lie algebras have been developed, which depend on families of free parameters. A common feature of the Hopf algebras obtained in [2] and [3] is that as algebras they are the same as the standard Drinfeld–Jimbo quantum groups [6]; the extra parameters only enter into the co-algebraic structures. The only known example which has a different algebraic structure from that of the corresponding one-parameter quantum supergroup is the two-parameter quantum sl(1/1) of [5]. In this letter we report on another such quantum supergroup associated with the Lie superalgebra osp(4/2) which, even as an algebra, depends on two free parameters q and p in a non-trivial way, thus intrinsically different from the standard $U_q(\mathrm{osp}(4/2))$ [7]. This two-parameter quantum supergroup also admits non-trivial finite-dimensional irreducible representations for generic q and p, and in appropriate limits of p, it reduces to $U_q(\mathrm{osp}(4/2))$ and $U_q(D(2,1;\alpha))$ respectively.

Recall that the Lie superalgebra osp(4/2) is of rank 3. Let α_i, $i=1,2,3$ be its simple roots with α_2 the odd one, and $H^*=\bigoplus_{i=1}^{3} C\alpha_i$. A bilinear form $(\cdot,\cdot)$ is defined on H^* such that

$$(\alpha_1,\alpha_1)=(\alpha_3,\alpha_3)=2 \qquad (\alpha_2,\alpha_2)=0$$
$$(\alpha_i,\alpha_j)=(\alpha_j,\alpha_i)=-(\delta_{ij+1}+\delta_{ij-1}) \qquad \forall i\neq j.$$

We define the two-parameter quantum supergroup $U_{q,p}(\mathrm{osp}(4/2))$ to be the Z_2-graded algebra generated by $\{e_i, f_i, h_i \mid i=1,2,3\}$ subject to the following relations:

$$\begin{aligned}
&[h_i, e_j]=(\alpha_i,\alpha_j)e_j \qquad [h_i, f_j]=-(\alpha_i,\alpha_j)f_j \qquad [h_i,h_j]=0 \qquad \forall, j\\
&[e_1,f_1]=\frac{q^{h_1}-q^{-h_1}}{q-q^{-1}} \qquad [e_3,f_3]=\frac{p^{h_3}-p^{-h_3}}{p-p^{-1}}\\
&\{e_2,f_2\}=\frac{q^{(d-h_1)/2}p^{(d-h_3)/2}-q^{-(d-h_1)/2}p^{-(d-h_3)/2}}{q-q^{-1}} \qquad [e_i,f_j\}=0 \qquad \forall i\neq j
\end{aligned} \tag{1}$$

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and

$$\begin{aligned}
&i^+ := [e_1, e_3] = 0 \qquad i^- := [f_1, f_3] = 0 \\
&i_2^+ := (e_2)^2 = 0 \qquad i_2^- = (f_2)^2 = 0 \\
&i_1^+ := (e_1)^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 (e_1)^2 = 0 \\
&i_3^+ := (e_3)^2 e_2 - (p + p^{-1}) e_3 e_2 e_3 + e_2 (e_3)^2 = 0 \\
&i_1^- := (f_1)^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 (f_1)^2 = 0 \\
&i_3^- := (f_3)^2 f_2 - (p + p^{-1}) f_3 f_2 f_3 + f_2 (f_3)^2 = 0
\end{aligned} \tag{2}$$

where $d = h_2 + (h_1 + h_3)/2$. Note that the denominator in the anticommutator $\{e_2, f_2\}$ is immaterial, as it can always be altered by multiplying e_2 and f_2 with scaler factors.

It is very important to observe that equation (1) alone enforces that the i^+'s (anti)commute with all the f's, and the i^-'s with all the e's, i.e.

$$[i^+, f_j] = [i_t^+, f_j] = 0 \qquad [i^-, e_j] = [i_t^-, e_j] = 0 \qquad \forall t, j = 1, 2, 3 \tag{3}$$

which guarantees the self-consistency of our definition of $U_{q,p}(\mathrm{osp}(4/2))$.

Co-algebraic structures can also be introduced into this two-parameter quantum supergroup in consistency with the algebraic structure. A possible co-multiplication is

$$\begin{aligned}
&\Delta(e_1) = e_1 \otimes q^{h_1} + 1 \otimes e_1 \qquad \Delta(f_1) = f_1 \otimes 1 + q^{-h_1} \otimes f_1 \\
&\Delta(e_3) = e_3 \otimes p^{h_3} + 1 \otimes e_3 \qquad \Delta(f_3) = f_3 \otimes 1 + p^{-h_3} \otimes f_3 \\
&\Delta(e_2) = e_2 \otimes q^{(d-h_1)/2} p^{(d-h_3)/2} + 1 \otimes e_2 \\
&\Delta(f_2) = f_2 \otimes 1 + q^{-(d-h_1)/2} p^{-(d-h_3)/2} \otimes f_2 \\
&\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \forall i
\end{aligned} \tag{4}$$

with the co-unit

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(h_i) = 0 \qquad \forall i \qquad \varepsilon(1) = 1$$

and antipode

$$\begin{aligned}
&S(e_1) = -e_1 q^{-h_1} \qquad S(f_1) = -q^{h_1} f_1 \\
&S(e_3) = -e_3 p^{-h_3} \qquad S(f_3) = -p^{h_3} f_3 \\
&S(e_2) = -e_2 q^{-(d-h_1)/2} p^{-(d-h_3)/2} \\
&S(f_2) = -q^{(d-h_1)/2} p^{(d-h_3)/2} f_2 \\
&S(h_i) = -h_i \qquad i = 1, 2, 3.
\end{aligned} \tag{5}$$

For later use we define

$$e_{\alpha_1+2\alpha_2+\alpha_3} = e_2 e_{\alpha_1+\alpha_2+\alpha_3} + (pq)^{-1} e_{\alpha_1+\alpha_2+\alpha_3} e_2 \tag{6}$$

$$f_{\alpha_1+2\alpha_2+\alpha_3} = f_{\alpha_1+\alpha_2+\alpha_3} f_2 + pq f_2 f_{\alpha_1+\alpha_2+\alpha_3} \tag{7}$$

with

$$\begin{aligned}
&e_{\alpha_2+\alpha_3} = e_2 e_3 - p^{-1} e_3 e_2 \qquad f_{\alpha_2+\alpha_3} = f_3 f_2 - p f_2 f_3 \\
&e_{\alpha_1+\alpha_2+\alpha_3} = e_1 e_{\alpha_2+\alpha_3} - q^{-1} e_{\alpha_2+\alpha_3} e_1 \\
&f_{\alpha_1+\alpha_2+\alpha_3} = f_{\alpha_2+\alpha_3} f_1 - q f_1 f_{\alpha_2+\alpha_3}.
\end{aligned}$$

Direct computations can easily establish that

$$[e_{\alpha_1+2\alpha_2+\alpha_3}, f_{\alpha_1+2\alpha_2+\alpha_3}] = \left(\frac{pq-(pq)^{-1}}{q-q^{-1}}\right)^2 \frac{(pq)^d-(pq)^{-d}}{pq-(pq)^{-1}}. \tag{8}$$

Highest weight irreps of this algebra at generic q and p can be constructed in the usual way by taking the e's as raising operators with the f's as lowering ones, and using the h's to label weights. Let $V(\Lambda)$ be an irreducible $U_{q,p}(\mathrm{osp}(4/2))$-module with the highest weight vector v_+^Λ. Define

$$h_1 v_+^\Lambda = \lambda_1 v_+^\Lambda \qquad h_3 v_+^\Lambda = \lambda_3 v_+^\Lambda \qquad d v_+^\Lambda = \lambda v_+^\Lambda. \tag{9}$$

A necessary condition for the irrep to be finite dimensional is that

$$\lambda_1, \lambda_3, -\lambda \in \mathbf{Z}_+ \tag{10}$$

where the condition on λ follows from the fact that $e_{\alpha_1+2\alpha_2+\alpha_3}, f_{\alpha_1+2\alpha_2+\alpha_3}$ together with $-d$ form a quantum sl(2) algebra. In fact it can be shown that for generic q and p there exists no finite dimensional irrep with $\lambda=0$ *or* -1, apart from the trivial one, and the simplest non-trivial irrep has a highest weight Λ such that $\lambda_1=\lambda_2=0$, $\lambda=-2$, which is 17-dimensional.

Let us now examine $U_{q,p}(\mathrm{osp}(4/2))$ in some particular limits of its deformation parameters.

(1) When $p=q$, the relations (1) and (2) coincide with the defining relations of the standard quantum osp(4/2), thus we have $U_{q,q}(\mathrm{osp}(4/2)) \simeq U_q(\mathrm{osp}(4/2))$.

(2) When $p=q^\alpha$ with $0 \neq \alpha \in \mathbf{C}$, we define

$$\widetilde{h_1} = h_1 \qquad \widetilde{h_3} = \alpha h_3 \qquad \widetilde{h_2} = [(1+\alpha)d - h_1 - \alpha h_2]/2.$$

Then equation (1) can be rewritten as

$$[\widetilde{h_i}, e_j] = (\widetilde{\alpha_i}, \widetilde{\alpha_j}) e_j \qquad [\widetilde{h_i}, f_j] = -(\widetilde{\alpha_i}, \widetilde{\alpha_j}) f_j$$

$$[\widetilde{h_i}, \widetilde{h_j}] = 0 \qquad [e_i, f_j\} = \delta_{ij} \frac{q^{\widetilde{h_i}} - q^{-\widetilde{h_i}}}{q_i - q_i^{-1}} \qquad \forall i,j \tag{11}$$

with $q_1=q_2=q$, $q_3=q^\alpha$. In (11) $\widetilde{\alpha_i}$, $i=1,2,3$ under the bilinear form $(\cdot\,,\cdot)$ have the same properties as the simple roots of the exceptional Lie superalgebra $D(2,1;\alpha)$, and (2) exactly reduces to the Serre relations for the quantum supergroup $U_q(D(2,1;\alpha))$. Hence $U_{q,q^\alpha}(\mathrm{osp}(4/2)) \simeq U_q(D(2,1;\alpha))$.

(3) When $p=q^{-1}$, we have

$$[e_{\alpha_1+2\alpha_2+\alpha_3}, f_i] = [f_{\alpha_1+2\alpha_2+\alpha_3}, e_i] = 0 \qquad \forall i. \tag{12}$$

Therefore we may set, in $U_{q,q^{-1}}(\mathrm{osp}(4/2))$,

$$e_{\alpha_1+2\alpha_2+\alpha_3} = f_{\alpha_1+2\alpha_2+\alpha_3} = 0. \tag{13}$$

Equations (1) and (2) supplemented by (13) define the quantum supergroup

$U_q(sl(2/2))$, i.e.,

$$U_q(sl(2/2)) \simeq U_{q,q^{-1}}(osp(4/2))/\langle(13)\rangle.$$

(4) Finally we look at the case with $p=1$ and q arbitrary. Now equations (1) and (2) reduce to

$$\begin{aligned}
&[h_i, e_j] = (\alpha_i, \alpha_j)e_j \qquad [h_i, f_j] = -(\alpha_i, \alpha_j)f_j \qquad [h_i, h_j] = 0 \qquad \forall i, j \\
&[e_1, f_1] = \frac{q^{h_1} - q^{-h_1}}{q - q^{-1}} \\
&\{e_2, f_2\} = \frac{q^{(d-h_1)/2} - q^{-(d-h_1)/2}}{q - q^{-1}} \\
&[e_3, f_3] = h_3 \qquad [e_i, f_j\} = 0 \qquad \forall i \neq j \\
&[e_1, e_3] = [f_1, f_3] = 0 \qquad (e_2)^2 = (f_2)^2 = 0 \\
&(e_1)^2 e_2 - (q + q^{-1})e_1 e_2 e_1 + e_2(e_1)^2 = 0 \\
&(f_1)^2 f_2 - (q + q^{-1})f_1 f_2 f_1 + f_2(f_1)^2 = 0 \\
&[e_3, [e_3, e_2]] = [f_3, [f_3, f_2]] = 0.
\end{aligned} \tag{14}$$

This defines a curious partially deformed algebra which contains as subalgebras a $U_q(sl(2/1))$ generated by $h_1, d, e_i, f_i, i = 1, 2$ and the universal enveloping algebra of an undeformed sl(2) generated by e_3, f_3, h_3.

It appears that all the other finite dimensional Lie algebras and Lie superalgebras of rank greater than 1 do not allow multiparameter quantizations of the kind reported here, namely, with both the algebraic and co-algebraic structures being multiparameter dependent and admitting finite dimensional irreps at generic deformation parameters. The superalgebra osp(4/2) turns out to be an exception presumably because of the existence of the finite dimensional Lie superalgebra $D(2, 1; \alpha)$, which is closely related to osp(4/2) and has a Cartan matrix depending on the complex parameter α. However, quantizations with the above-mentioned properties but involving only one generic complex parameter q and an Nth primitive root of unity may exist for the other Lie algebras, which ought to give rise to the standard one-parameter quantum groups and quantum supergroups when N equals 1 and 2 respectively.

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References

[1] Manin I Yu 1989 *Commun. Math. Phys.* **123** 163

[2] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697

[3] Reshetikhin N Yu 1991 *Lett. Math. Phys.* **20** 331

[4] Fairlie D B and Zackos C K 1991 *Phys. Lett.* **256B** 43
Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317
Schirrmacher A 1991 *Z. Phys.* C **50** 320

[5] Hinrichsen H and Rittenberg V 1992 The Pokrovski-Talapov phase transition and quantum groups *Preprint* CERN-TH. 6411/92

[6] Drinfeld V G 1986 Quantum groups *Proc. ICM (Berkeley)* p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63

[7] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
Zhang R B, Bracken A J and Gould M D 1991 *Phys. Lett.* **257B** 133
Floreanini R, Spiridonov V P and Vinet L 1991 *Commun. Math. Phys.* **137** 149
Zhang R B 1991 Finite dimensional irreducible representations of the quantum supergroup $U_q(gl(m/n))$ *Preprint* Australian National University
Vinet L *et al* 1991 *Preprint* University of Montreal UdeM-LPN-TH53
Scheunert M 1991 *Preprint* University of Bonn BOON-HE-91-10